Reduced differential transform method for solving partial differential equations with variable coefficients

Hassan Hosseinzadeh¹, Elham Salehpour²

1,2. Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran Sciences, University of Mazandaran, Babolsar, Iran
1. Department of Mathematics, Qaemshar Branch, Islamic Azad University, Qaemshar, Iran
2. Department of Mathematics, Nowshahr Branch, Islamic Azad University, Nowshahr, Iran

Corresponding author email: Elham_salehpor61@yahoo.com

ABSTRACT: In this paper, we consider the reduced differential transform method (RDTM) for finding approximate and exact solutions of some partial differential equations with variable coefficients. The efficiency of the considered method is illustrated by some examples. The results reveal that the proposed method is very effective and simple and can be applied for other linear and nonlinear problems in mathematical physics. Moreover, particular examples are discussed to show reliability and the performance of the modified decomposition method.

Keywords: Reduced differential transform method; Partial differential equations variable coefficients.

INTRODUCTION

The differential transformation method is a numerical method based on a Taylor expansion. This method constructs an analytical solution in the form of a polynomial. The concept of differential transform method was first proposed and applied to solve linear and nonlinear initial value problems in electric circuit analysis by [1]. Chen and Liu have applied this method to solve two-boundary-value problems [2]. Jang, Chen and Liu apply the two-dimensional differential transform method to solve partial differential equations [3]. Yu and Chen apply the differential transformation method to the optimization of the rectangular fins with variable thermal parameters [4,5].

The differential transform method (DTM) is a numerical as well as analytical method for solving integral equations, ordinary, partial differential equations and differential equation. The method provides the solution in terms of convergent series with easily computable components. The concept of the differential transform was first proposed by Zhou [1] and its main application concern with both linear and nonlinear initial value problems in electrical circuit analysis. The DTM gives exact values of the nth derivative of an analytic function at a point in terms of known and unknown boundary conditions in a fast manner. This method constructs, for differential equations, an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic computations of the necessary derivatives of the data functions. The Taylor series method is computationally taken long time for large orders. The DTM is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. Different applications of DTM can be found in [6–25].

By using these methods, differential equations are transformed into algebraic equations which are easier to cope with. In fact, transform methods are more complex and difficult when applying to nonlinear problems. A different dealing method to solve non-linear initial value problems is the RDTM. Our motivation is to concentrate on the applications of the reduced differential transform method (RDTM). It should be mentioned that one of the main advantages of the RDTM is its ability in providing us a continuous representation of the approximate solution, which allows better information of the solution over the time interval.

The reduced differential transform method (RDTM) will be employed in a straightforward manner without any need of linearization or smallness assumptions. DTM was first applied in the engineering domain. DTM provides an efficient explicit and numerical solution with high accuracy, minimal calculations, sparing of physically unrealistic assumptions. However, RDTM has some drawbacks. By using RDTM, we obtain a series
solution, in practice a truncated series solution. This series solution does not exhibit the periodic behavior which is characteristic of oscillator equations and gives a good approximation.

REDUCED DIFFERENTIAL TRANSFORM METHOD

To illustrate the basic idea of the DTM, we considered \( u(x, t) \) is analytic and differentiated continuously in the domain of interest, then let

\[
U_k(x) = \frac{1}{k!} \left( \frac{d^k u(x, t)}{dt^k} \right)_{t=t_0},
\]

where the spectrum \( U_k(x) \) is the transformed function, which is called T-function in brief. The differential inverse transform of \( U_k(x) \) is defined as follows:

\[
u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k u(x, t)}{dt^k} \right)_{t=t_0} (t-t_0)^k,
\]

combining (3) and (4), it can be obtained that

\[
u(x, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{d^k u(x, t)}{dt^k} \right)_{t=t_0} t^k,
\]

and Equation (4) is shown as

\[
u(x, t) = \sum_{k=0}^{\infty} U_k(x) t^k,
\]

usually, the values of \( n \) is decided by convergence of the series coefficients.

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion. The fundamental operations of reduced differential transform method are listed in Table 1 below.

<table>
<thead>
<tr>
<th>Original function</th>
<th>Transformed function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u(x, t) )</td>
<td>( U_k(x) = \frac{1}{k!} \left( \frac{d^k u(x, t)}{dt^k} \right)_{t=t_0} )</td>
</tr>
<tr>
<td>( w(x, t) = u(x, t) + v(x, t) )</td>
<td>( W_k(x) = U_k(x) + V_k(x) )</td>
</tr>
<tr>
<td>( w(x, t) = a u(x, t) )</td>
<td>( W_k(x) = a U_k(x) )</td>
</tr>
<tr>
<td>( w(x, t) = x^n t^m )</td>
<td>( W_k(x) = x^n U_{k-n}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = x^m u(x, t) )</td>
<td>( W_k(x) = \sum_{r=0}^{k} \binom{k}{r} U_r(x) V_{k-r}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial}{\partial t} u(x, t) )</td>
<td>( W_k(x) = (k+1)...(k+r) U_{k+r}(x) )</td>
</tr>
<tr>
<td>( w(x, t) = \frac{\partial}{\partial x} u(x, t) )</td>
<td>( W_k(x) = \frac{\partial}{\partial x} U_k(x) )</td>
</tr>
</tbody>
</table>

Applications

To illustrate the effectiveness of the present method, several examples are considered in this section. The accuracy of the method is assessed by comparison with the exact solutions.

Example 1. Solve the following partial differential equation[27]:

\[
(x^2 - x - 2) \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} - (2x - 1) \frac{\partial^2 u}{\partial y \partial x} = 0,
\]

with the initial values

\[
u(x, 0) = x, \quad \frac{\partial u(x, 0)}{\partial y} = 1
\]

Solution. Taking the reduced differential transformation of Eq. (7), then

\[
(x^2 - x - 2)(k+1)(k+2) U_{k+2}(x) = -\frac{\partial^2}{\partial x^2} U_k(x) + (2x - 1) \frac{\partial}{\partial x} (k+1) U_{k+1}(x)
\]

From the initial condition given by Eq. (8)

\[
U_0(x) = x, \quad U_1(x) = 1
\]

\[
U_2(x) = 0, \quad U_3(x) = 0
\]

...
Hence, the exact solution \[ (14) \]
\[ u(x, y) = x + y. \]
And the differential transformation of Eq. (7), is too.\[ (15) \]

Example 2. Solve the Partial differential equation \[ (26) \]
\[ u_{tt} - u_{xx} - u = 0, \] with the initial values
\[ u(x, 0) = 1 + \sin x, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \] (17)
Solution. Taking the reduced differential transformation of Eq. (16), then
\[ (k + 1)(k + 2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + U_k(x) \] (18)
From the initial condition given by Eq. (17)
\[ U_0(x) = 1 + \sin x, \] (19)
\[ U_1(x) = 0, \] (20)
\[ \rightarrow U_2(x) = \frac{1}{2!} \] (21)
\[ \rightarrow U_3(x) = 0, \] (22)
\[ \rightarrow U_4(x) = \frac{1}{4!} \] (23)
\[ \rightarrow U_5(x) = 0, \] (24)
\[ \rightarrow U_6(x) = \frac{1}{6!} \] (25)
\[ \vdots \]
\[ u(x, t) = \sin x + 1 + \frac{\sin x}{2!} + \frac{t^2}{4!} + \frac{t^4}{6!} + \cdots = \sin x + \cosh t \] (26)
In this case, we have obtained the exact solution of the targeted equation with the specified initial conditions.

Example 3. Consider the nonlinear non-homogeneous equation:
\[ (27) \]
\[ u_{tt} - u_{xx} - 2u = -2\sin x \sin t \]
suppose to the initial values
\[ u(x, 0) = 0, \quad u_t(x, 0) = \sin x \] (28)
The reduced differential transformation of Eq. (27) is
\[ (k + 1)(k + 2)U_{k+2}(x) = \frac{\partial^2}{\partial x^2}U_k(x) + 2U_k(x) - 2(\sin x) \frac{1}{k!} \sin \left(\frac{\pi}{2}k\right) \] (29)
the transformed initial conditions are
\[ U_0(x) = 0, \] (30)
\[ U_1(x) = \sin x, \] (31)
\[ U_2(x) = -\frac{1}{3!} \sin x, \] (32)
\[ U_3(x) = 0, \] (33)
\[ U_4(x) = \frac{1}{5!} \sin x, \] (34)
\[ \vdots \]
we obtained the closed form solution as
\[ u(x, t) = \sin x \sin t. \] (36)
Which is the exact solution of (27) and solution with DTM \[ (26) \] for Eq. (27)

\section*{CONCLUSION}

The reduced differential transform method has been successfully applied for solving Partial differential equations with variable coefficients. The solution obtained by reduced differential transform method is an infinite power series for appropriate initial condition, which can in turn express the exact solutions in a closed form. The results show that the reduced differential transform method is a powerful mathematical tool for solving partial differential equations with variable coefficients. Thus, we conclude that the proposed method can be extended to solve many PDEs with variable coefficients which arise in physical and engineering applications.

\section*{REFERENCES}