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Exact analytical solution to a nonlinear oscillator typified by a mass attached to a stretched wire

Elyas Shivanian[1], Mahdi Anbarloei[2]

Department of Mathematics, Imam Khomeini International University, Qazvin, 34149-16818, Iran

* ***Corresponding author email:*** shivanian@sci.ikiu.ac.ir

ABSTRACT: This article shows that the well known nonlinear oscillator of a mass attached to a stretched elastic wire, investigated recently by some numerical and semi-analytical approximative methods, is exactly solvable and furthermore, gives analytical exact solution in terms of a new non-algebraic function namely $\mathbb{S}\mathrm{im}$ function for further physical interpretation. It is revealed that $\mathbb{S}\mathrm{im}$ function is well-defined, differentiable and decreasing function on its domain. It is shown graphically some exact related results.
Key words : Exact analytical solutions; $\mathbb{S}\mathrm{im}$ function; Nonlinear oscillator.

Preliminaries and problem formulation

The governing ordinary differential equation of motion and the associated initial conditions for a mass attached to a stretched elastic wire (Mickens, 1996), (Durmaza et al., 2011) as shown in Figure 1, are:

$$m\frac{d^2x}{d\tau^2} = -2T\sin\theta, \tag{1}$$
$$x(0) = X, \tag{2}$$
$$\frac{dx}{d\tau}(0) = 0, \tag{3}$$

where T is the tension on the elastic wire after stretching and X is the initial amplitude. The tension T can be expressed in terms of the initial tension T_0 and the axial rigidity of the elastic wire K as follows

$$T = T_0 + K\frac{\sqrt{L^2+x^2}-L^2}{L^2}. \tag{4}$$

Using Eq. (4), Eq. (1) can be rewritten as

$$m\frac{d^2x}{d\tau^2} + 2K\frac{x}{L} + \frac{\frac{x}{L}}{\sqrt{1+\left(\frac{x}{L}\right)^2}}(2T_0 - 2K) = 0. \tag{5}$$

Introducing the following dimensionless quantities

$$y = \frac{x}{L}, \qquad t = \frac{\tau}{\sqrt{\frac{mL}{2K}}}, \tag{6}$$

and substituting into Eq. (5), one can derive the nondimensional form of the equation (5), i.e.

$$\frac{d^2y}{dt^2} + y - \left(1 - \frac{T_0}{K}\right)\frac{y}{\sqrt{1+y^2}} = 0, \quad 0 \le T_0 \le K. \tag{7}$$

Upon using $\lambda = \left(1 - \frac{T_0}{K}\right)$, Eq. (7) yields

$$\frac{d^2y}{dt^2} + y - \frac{\lambda y}{\sqrt{1+y^2}} = 0, \quad 0 \le \lambda \le 1, \tag{8}$$

with initial conditions

$$u(0) = A, \tag{9}$$
$$\frac{dy}{dt}(0) = 0, \tag{10}$$

which come from original initial conditions (2)-(3), where $A = \frac{X}{L}$ is amplitude of the system.

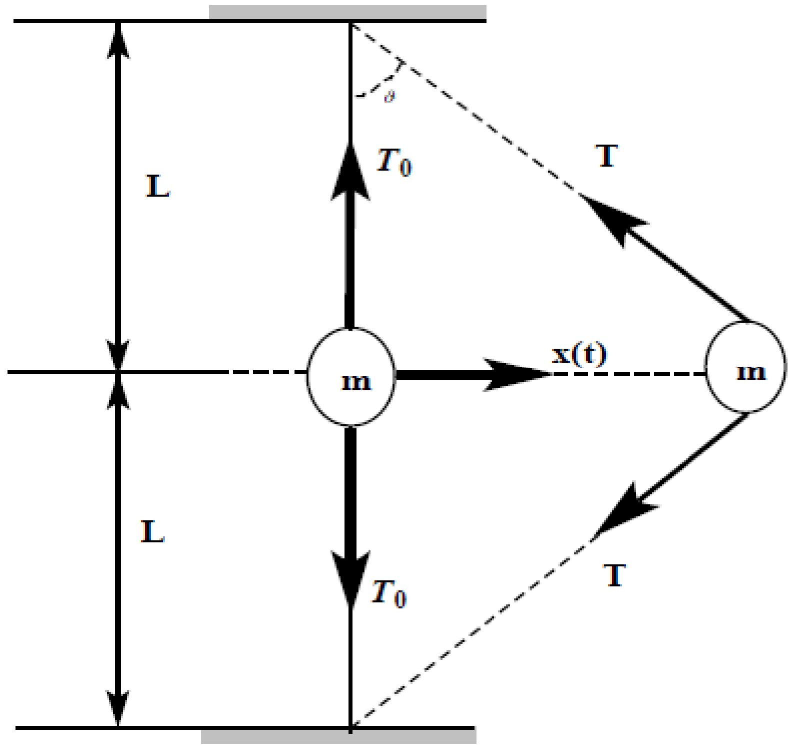

Figure 1. Mass attached to a stretched elastic wire.

The problem (8)-(10) has been recently considered and analyzed by some numerical and semi-analytical techniques. As a brief review we can mention: He's max-min approach (Durmaza et al., 2011), homotopy perturbation method (Beléndez et. al., 2009) piecewise variational iteration method (Geng, 2011), generalized Senator-Bapat perturbation technique (Lai et. Al., 2010), optimal homotopy asymptotic method (OHAM) (Golbabai), (Liu et. Al., 2013) harmonic balance method (Sun et. al., 2007), (Beléndez et. al. 2007) He's parameter-expansion method (Xu, 2007) and many references therein.

$\mathbb{S}im$ function and its properties

Definition 1 Let $0 \le \lambda \le 1$, we define the function $\mathbb{S}\mathrm{im}: [-A, A] \to \mathbb{R}^+$ as

$$\mathbb{S}\mathrm{im}(z) = \int_z^{|A|} \frac{d\tau}{\sqrt{A^2+2\lambda\sqrt{1+\tau^2}-\tau^2-2\lambda\sqrt{1+A^2}}}. \quad (11)$$

Lemma 1 If $0 \le \lambda \le 1$, $|\tau| \le |A|$ then the expression

$$A^2 + 2\lambda\sqrt{1+\tau^2} - \tau^2 - 2\lambda\sqrt{1+A^2}$$

is nonnegative.

Proof. It is clear that

$$\sqrt{1+A^2} + \sqrt{1+\tau^2} \ge 2 \ge 2\lambda. \quad (12)$$

Since $|\tau| \le A$, then $\sqrt{1+A^2} - \sqrt{1+\tau^2} \ge 0$. Now, multiplying both sides of Eq. (12) by $\sqrt{1+A^2} - \sqrt{1+\tau^2}$, we obtain

$$A^2 - \tau^2 \ge 2\lambda\sqrt{1+A^2} - 2\lambda\sqrt{1+\tau^2},$$

which yields

$$A^2 + 2\lambda\sqrt{1+\tau^2} - \tau^2 - 2\lambda\sqrt{1+A^2} \ge 0,$$

and this proves the Lemma 1.

Theorem 1 $\mathbb{S}\mathrm{im}$ as a function of z as defined in Definition 1, is well-defined, nonnegative, differentiable (thus it is continuous) and decreasing function.

Proof. From the Lemma 1, the denominator of integrand in $\mathbb{S}\mathrm{im}$ function is always positive in its domain i.e. $z \in [-A, A]$, therefore the $\mathbb{S}\mathrm{im}$ function is nonnegative. In fact, it defines a function from $[-A, A]$ to $\mathbb{R}^+$. If

$\mathbb{S}\mathrm{im}(z_1) = \mathbb{S}\mathrm{im}(z_2)$ then

$$\int_{z_1}^{A} \frac{d\tau}{\sqrt{A^2+2\lambda\sqrt{1+\tau^2}-\tau^2-2\lambda\sqrt{1+A^2}}} = \int_{z_2}^{A} \frac{d\tau}{\sqrt{A^2+2\lambda\sqrt{1+\tau^2}-\tau^2-2\lambda\sqrt{1+A^2}}},$$

we again mention that the integrand is positive so $z_1 = z_2$ and $\mathbb{S}\mathrm{im}(z)$ is well-defined. It can be seen easily that $\mathbb{S}\mathrm{im}(z)$ is differentiable and more

$$\frac{d\mathbb{S}\mathrm{im}(z)}{dz} = -\frac{1}{\sqrt{A^2+2\lambda\sqrt{1+\tau^2}-\tau^2-2\lambda\sqrt{1+A^2}}} < 0$$

therefore $\mathbb{S}\mathrm{im}(z)$ is decreasing function, and the proof is completed.

Corollary 1 By the change of variable

$$\eta(\tau) = \eta = \sqrt{1+\tau^2} - \lambda$$

in Definition 1, one can obtain

$$\mathbb{S}\mathrm{im}(z) = \int_{\sqrt{1+z^2}-\lambda}^{\sqrt{1+A^2}-\lambda} \frac{\eta+\lambda}{\sqrt{B^2-\eta^2}\sqrt{(\eta+\lambda)^2-1}}\, d\eta, \quad (13)$$

where $B = \eta(A) = \sqrt{1+A^2} - \lambda$. The last integral is expressible (e.g. via Wolfram's Mathematica) up to a constant term as a linear combination of elliptic integrals of the first kind and, complete elliptic integrals of the first and third kind i.e.

$$\mathbb{S}\mathrm{im}(z) = \frac{\left(\begin{array}{l} 2\left(\lambda+\sqrt{z^2+1}-1\right)\left(\lambda+\sqrt{z^2+1}+1\right)\sqrt{-\frac{B+\sqrt{z^2+1}}{(B-\lambda-1)\left(\lambda+\sqrt{z^2+1}-1\right)}}\sqrt{\frac{\sqrt{z^2+1}-B}{(B+\lambda+1)\left(\lambda+\sqrt{z^2+1}-1\right)}}\times \\ \left(\mathrm{EllipticF}\left(\mathrm{Arcsin}\left(\sqrt{\frac{(B+\lambda-1)\left(\lambda+\sqrt{z^2+1}+1\right)}{(B+\lambda+1)\left(\lambda+\sqrt{z^2+1}-1\right)}}\right), \frac{(B-\lambda+1)(B+\lambda+1)}{(B-\lambda-1)(B+\lambda-1)}\right)\right. \\ \left. -2\mathrm{EllipticPi}\left(\frac{B+\lambda+1}{B+\lambda-1}, \mathrm{Arcsin}\left(\sqrt{\frac{(B+\lambda-1)\left(\lambda+\sqrt{z^2+1}+1\right)}{(B+\lambda+1)\left(\lambda+\sqrt{z^2+1}-1\right)}}\right), \frac{(B-\lambda+1)(B+\lambda+1)}{(B-\lambda-1)(B+\lambda-1)}\right)\right) \end{array}\right)}{\sqrt{\frac{(B+\lambda-1)\left(\lambda+\sqrt{z^2+1}+1\right)}{(B+\lambda+1)\left(\lambda+\sqrt{z^2+1}-1\right)}}\sqrt{(B^2-z^2-1)\left(\lambda\left(\lambda+2\sqrt{z^2+1}\right)+z^2\right)}}$$
$$-\frac{\left(\begin{array}{l} 2(B+\lambda-1)(B+\lambda+1)\sqrt{-\frac{B}{B^2-2B-\lambda^2+1}}\sqrt{\frac{1}{B^2+2B\lambda+\lambda^2-1}}\times \\ \left(\mathrm{EllipticK}\left(\frac{(B-\lambda+1)(B+\lambda+1)}{(B-\lambda-1)(B+\lambda-1)}\right) - 2\mathrm{EllipticPi}\left(\frac{B+\lambda+1}{B+\lambda-1}, \frac{(B-\lambda+1)(B+\lambda+1)}{(B-\lambda-1)(B+\lambda-1)}\right)\right) \end{array}\right)}{\sqrt{-B(B^2+2B\lambda+\lambda^2-1)}}. \quad (14)$$

We have plotted $\mathbb{S}\mathrm{im}(z)$ in its domain $[-A, A]$ for different values of λ and A, in Figures 2 and 3, respectively.

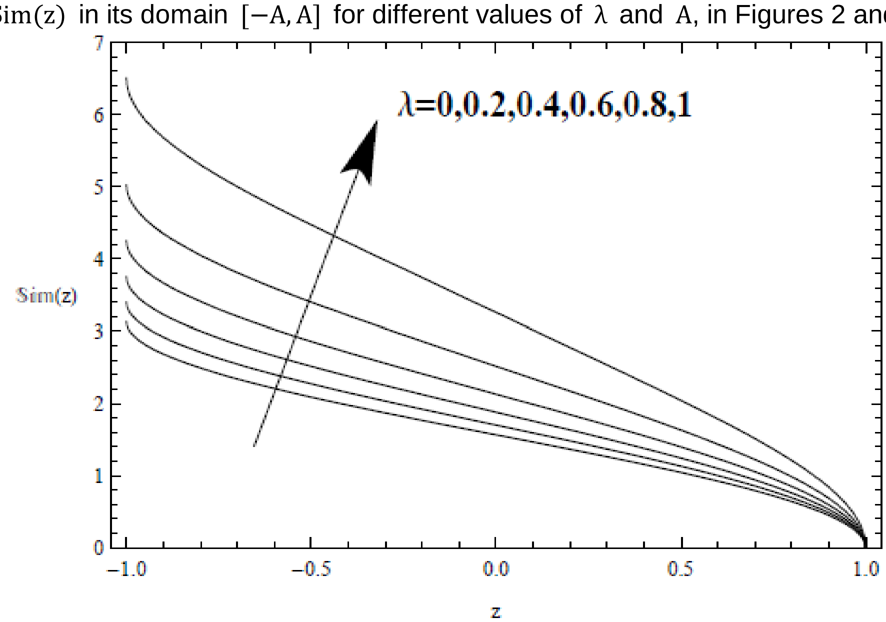

Figure 2. Diagram of the function $\mathbb{S}im(z)$ for different values of λ.

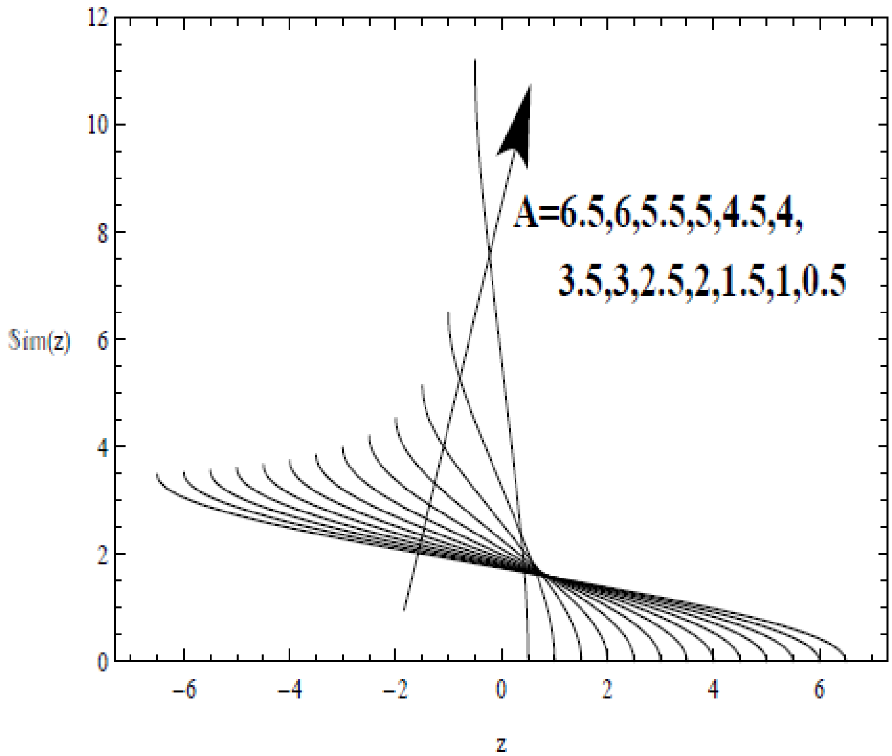

Figure 3. Diagram of the function $\mathbb{S}im(z)$ for different values of A.

The exact analytical solution

One easily sees that Eq. (8) admits the first integral

$$\frac{1}{2}\left(\frac{dy}{dt}\right)^2+\frac{1}{2}y^2-\lambda\sqrt{1+y^2}=C, \tag{15}$$

where C is an integration constant. The conditions (9) and (10) give the integration constant C as

$$C=\frac{1}{2}A^2-\lambda\sqrt{1+A^2}. \tag{16}$$

Using above C, Eq. (15) can be rewritten as

$$\left(\frac{dy}{dt}\right)^2=-2\lambda\sqrt{1+A^2}+A^2-y^2+2\lambda\sqrt{1+y^2}, \tag{17}$$

or equivalently

$$\frac{dy}{dt}=\sqrt{-2\lambda\sqrt{1+A^2}+A^2-y^2+2\lambda\sqrt{1+y^2}}. \tag{18}$$

Eq. (18), by applying the condition (9), yields

$$t=\int_y^A\frac{d\tau}{\sqrt{A^2+2\lambda\sqrt{1+\tau^2}-\tau^2-2\lambda\sqrt{1+A^2}}}. \tag{19}$$

Now, using the Definition (1), the exact analytical solution of the problem (8)-(10) is given by

$$t=\mathbb{S}im(y;\lambda,A), \quad 0\le\lambda\le 1, A\ge 0. \tag{20}$$

RESULTS AND DISCUSSIONS

The main advantages of the exact analytical solution (20) are the facts that firstly the mathematical properties of this non-algebraic functions has been well established in the section 2 and secondly, today well-performing computer software programs such as Mathematica and Maple are available both for symbolic and numerical calculations involving this function (in this paper Wolfram's Mathematica has been used) (Abbasbandy et. al., 2010), (Abbasbandy et. al., 2011), (Abbasbandy et. al., 2012), (Magyari 2008), (Magyari 2010).

Dependence of solution to the amplitude of the system

This subsection discusses the dependence of the system on the amplitude A when other parameter i.e. λ is fixed. In Figure 4, the dependence of the solution $y(t)$ on A has been plotted for ten different values of the

amplitude according to (20). An inspection of Figure 4 emphasizes three remarkable features, namely:
$y(t)$ as solution of the system is periodic function and oscillates between $-A$ and A.
The period of function $y(t)$ decreases when the amplitude A increases.
The number of oscillations increases while the amplitude A increases during a fixed period of time.

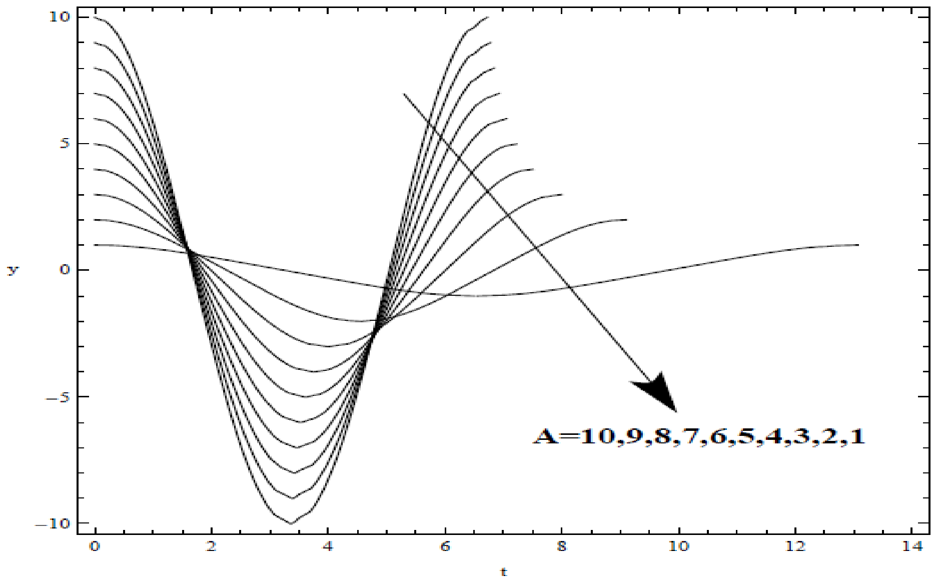

Figure4. Diagram of the one period of the solutions $y(t)$ for different values of A with $\lambda = 1$.

Dependence of solution to the parameter λ

To check the reliability of the solution to the parameter λ, we consider (20) when the amplitude A is fixed and plot y as a function of t for six different values of λ (Figure 5). Outcome of first insight to this Figure are
$y(t)$ as solution of the system is periodic function and oscillates between $-A$ and A.
The period of function $y(t)$ increases when the parameter λ increases.
The number of oscillations decreases while the parameter λ increases during a fixed period of time.
When $\lambda = 0$, it is clear that $\mathbb{S}im(y; 0, A) = arccos(\frac{y}{A})$. Therefore the solution (20) is simplified to $y = Acos(t)$, this is in full agreement with the Figure 5 in the case $\lambda = 0$.

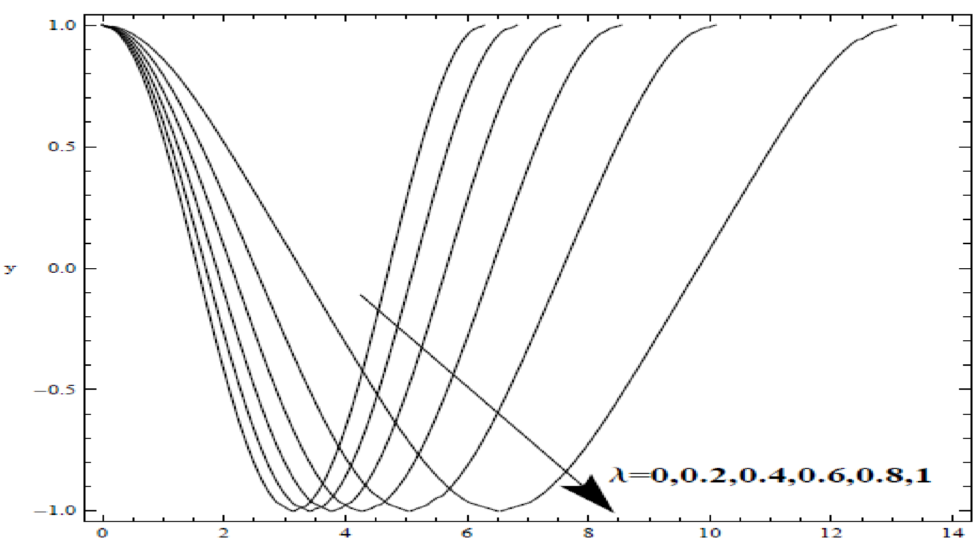

Figure5. Diagram of the one period of the solutions $y(t)$ for different values of λ with $A = 1$.

The exact angular frequency of the system

The exact angular frequency of the system ω is obtained easily from the exact solution (20). Setting $y = 0$ in this equation, we obtain

$$\frac{T}{4} = \mathbb{S}im(0; \lambda, A), \quad (21)$$

where T is the period of the system. It is a fact that $\omega = \frac{2\pi}{T}$, thus the Eq. (21) yields

$$\omega = \frac{\pi}{2\mathbb{S}im(0;\lambda,A)}. \quad (22)$$

The exact angular frequency (22) as a function of amplitude A has been shown for a different family of λ in Figure 6. As it is seen in this Figure, for a fixed λ, the angular frequency tends to one while the amplitude A goes to infinity.

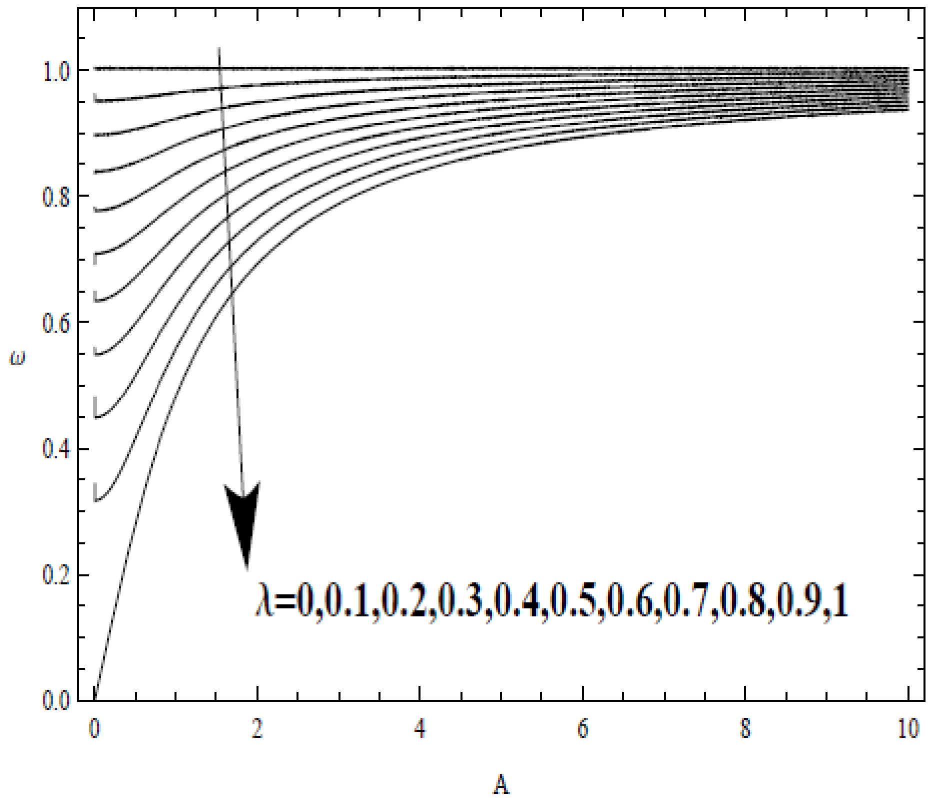

Figure 6. The exact angular frequency as a function of amplitude A for different values of λ.

The form of the limit cycles for system

In order to study the limit cycles of (8)-(10) with the time variable being implicit, it is convenient to consider it in the coordinates $(y, \frac{dy}{dt}) = (y, u)$ in the plane, with $u(y) = \frac{dy}{dt}$ and $\frac{d^2y}{d^2t} = u\frac{du}{dy}$. Therefore Eq. (17) is converted to the following

$$u^2 = -2\lambda\sqrt{1 + A^2} + A^2 - y^2 + 2\lambda\sqrt{1 + y^2}. \quad (23)$$

A limit cycle $C_l \equiv (y, u_{\pm}(y))$ of Eq. (23) has a positive branch $u_+(x) > 0$ and a negative branch $u_-(x) < 0$. They cut the horizontal axis in two points $(-A, 0)$ and $(A, 0)$ because the origin $(0,0)$ is the only fixed point of integration of Eq. (23) i.e.

$$uu' + y - \lambda\frac{y}{\sqrt{1+y^2}} = 0. \quad (24)$$

Then every limit cycle C_l solution of Eq. (23) encloses the origin and the oscillation y runs in the interval $-A \leq y \leq A$. The amplitudes of oscillation A identify the limit cycle. The result is a nested set of closed curves that defines the qualitative distribution of the integral curves in the plane (y, u). The exact limit cycles have been shown for different λ and A in Figures 7 and 8, respectively.

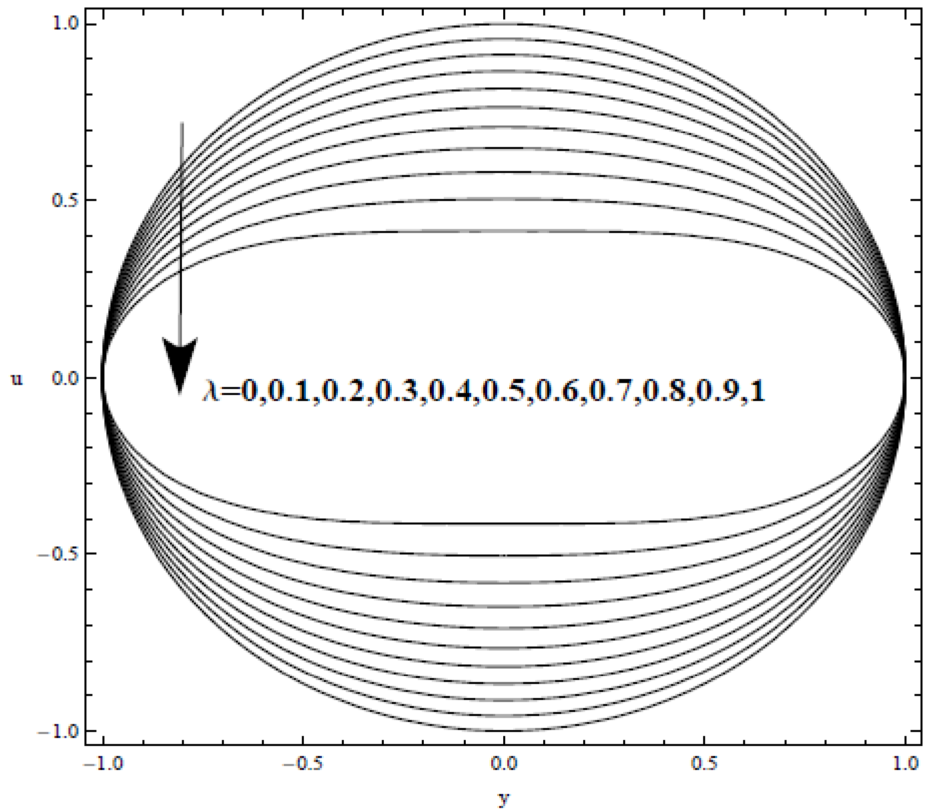

Figure 7. The exact limit cycles of the system (8)-(10) for different values of λ when $A = 1$.

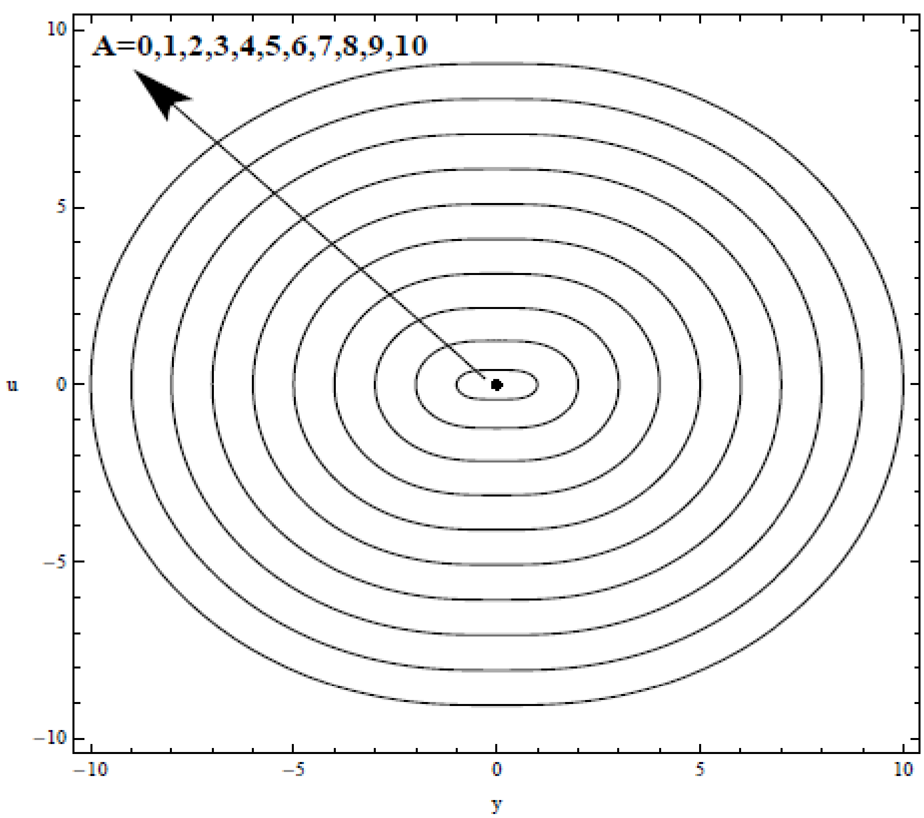

Figure8. The exact limit cycles of the system (8)-(10) for different values of A when $\lambda = 1$.

CONCLUSIONS

The mathematical model of a nonlinear oscillator described as a mass attached to a stretched elastic wire has been revisited. We have defined non-algebraic $\mathbb{S}\mathrm{im}$ function and established it's properties. The $\mathbb{S}\mathrm{im}$ function is well-defined, differentiable and decreasing function on it's domain. It has been proved that the exact analytical solution of the mentioned model can be expressed as $\mathbb{S}\mathrm{im}$ function. Consequently, we have shown graphically some exact related results for further physical interpretations.

REFERENCES

Abbasbandy S, Shivanian E, Hashim I. 2011. Exact analytical solution of forced convection in a porous-saturated duct, Commun. Nonlinear Sci. Numer. Simul., Vol. 16(10) pp. 3981–3989.

Abbasbandy S, Shivanian E. 2010. Exact analytical solution of a nonlinear equation arising in heat transfer, Phys. Lett. A, Vol. 374(4) pp. 567–574.

Abbasbandy S, Shivanian E. 2012. Exact analytical solution of the MHD Jeffery-Hamel flow problem, Meccanica, Vol. 47 pp. 1379–1389.

Beléndez A, Beléndez T, Neipp C, Hernández A, Álvarez ML. 2009. Approximate solutions of a nonlinear oscillator typified as a mass attached to a stretched elastic wire by the homotopy perturbation method, Chaos, Solitons & Fractals, Vol. 39 pp. 746–764.

Beléndez A, Hernández A, Beléndez T, Álvarez ML, Gallego S, Ortuño CNM. 2007. Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire, J. Sound Vib., Vol. 302 pp. 1018–1029.

Beléndez A, Pascual C, Méndez D, Neipp C. 2007. Solution of the relativistic (an)harmonic oscillator using the harmonic balance method, J. Sound Vib. Vol. 302, pp. 1018–1029.

Durmaza S, Demirbaoğ SA, Kayaa MO. 2011. Approximate solutions for nonlinear oscillation of a mass attached to a stretched elastic wire, Comput. Math. Appl., Vol. 61, pp. 578–585.

Geng F. 2011. A piecewise variational iteration method for treating a nonlinear oscillator of a mass attached to a stretched elastic wire, Comput. Math. Appl., Vol. 62 pp. 1641–1644.

Golbabai A, Fardi M, Sayevandc K. Application of the optimal homotopy asymptotic method for solving a strongly nonlinear oscillatory system, Math. Comput. Model. in press.

Lai SK, Xiang Y. 2010. Application of a generalized Senator-Bapat perturbation technique to nonlinear dynamical systems with an irrational restoring force, Comput. Math. Appl., Vol. 60 pp. 2078–2086.

Liu YP, Liao SJ, Li ZB. 2013. Symbolic computation of strongly nonlinear periodic oscillations, J. Symb. Comput., Vol. 55 pp. 72–95.

Magyari E. 2008. Exact analytical solution of a nonlinear reaction-diffusion model in porous catalysts, Chem. Eng. J., Vol. 143 pp. 167–171.

Magyari E. 2010. Exact analytical solutions of diffusion reaction in spherical porous catalyst, Chem. Eng. J., Vol. 158(2) pp. 266–270.

Mickens R. 1996. Oscillations in Planar Dynamic Systems, World Scientific Publishing, Singapore.

Sun WP, Wu BS, Lim CW. 2007. Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire, J. Sound Vib., Vol. 300, pp. 1042–1047.

Xu L. 2007. Application of He's parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire, Phys. Lett. A, Vol. 368 pp. 259–262.